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Semiclassical theory of Compton and photoelectric effects

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Abstract. Scattering of a plane electromagnetic wave by free and bound wave packets is analyzed by semiclassical radiation theory. It is shown that the theory gives the correct answer to the question of radiation intensity in the photoelectric effect and to the correlation problem in Compton scattering. The expression for the intensity of the scattered radiation differs from the cross section which is derived from the model based on the particle nature of the electromagnetic field. The meaning of this difference is discussed. Low frequency spectrum of the scattered radiation on a bound charge is obtained.

PACS. 03.65.Sq Semiclassical theories and applications – 32.80.Cy Atomic scattering, cross sections, and form factors; Compton scattering

1 Introduction

Scattering of an electromagnetic (EM) wave by a charge is an old problem and has been treated on numerous occasions. Two typical situations are encountered: (i) scattering by a free and (ii) by a bound charge. In the first case the Compton effect is dominant and its description for a delocalized, plane wave, charged particle is summarized in the Klein-Nishina formula [1]. This formulation gives the correct relationship between the initial and the final parameters of the scattering event (the frequency of the EM wave and momentum of the charge) and the correct intensity distribution of the scattered radiation (cross section). In the second case several effects are produced which are distinguished according to the nature of energy transfer between the EM field and the charge. Among those the Compton and the photoelectric effects are the best known, and their explanation is normally given through the concept of the photon. According to this the photoelectric effect happens when the energy of the scattered charge is equal to the energy of the absorbed photon (modified by the ionization energy), meaning that as a consequence no scattered photon (radiation) is observed. In the Compton effect, on the other hand, the scattered photon (radiation) is observed but its frequency is shifted relative to the frequency of the incident photon (EM wave). The kinematics of the Compton effect, both for a free and a bound charge, is very similar and straightforward. However, calculation of the cross section, whether the charge is free but localized (instead of a plane wave it is represented by a wave packet) or bound, is not straightforward. One of the earliest model for this purpose was developed by DuMond [2], and it was based on the assumption that the photon is scattered by a charge having a certain velocity. In the situation when the charge has a spread of velocities, as in a wave packet or in a bound state, the cross section is then directly related to the probability of finding a charge with this velocity (given by the squared modulus of the momentum space wave function).

A more rigorous (non relativistic) derivation of the cross section is again based on the assumption of the particle nature of the EM field [3–7]. The theory which is used for this purpose is generalized from the one for calculating cross sections in particle collisions, say collision of a charged particle with an atom (for a concise but a very instructive review see Ref. [8]). This is done by replacing the Hamiltonian for the incoming particle with the Hamiltonian representing a free photon, and the interaction term between the incoming and the target particle (atom) is replaced by the appropriate one when the photon is involved [9–11]. With these modifications one derives (usually in the form of a Born expansion) the photon-atom (Compton) cross section. From the general result various approximate ones are obtained, *e.g.* in the impulsive model one gets the DuMond cross section.

There has been, and still are, attempts to find an alternative description of the Compton and the photoelectric effects. One can think of several reasons why is this so, but the one which motivates the most is that the concept of the photon, and its theoretical consequences in the form of the quantum field theory, is not physically and mathematically a satisfying idea. By trying entirely standard classical ideas is perhaps too extreme view, but a viable compromise could be the semiclassical theory. By the standard classical theory it is meant: description of particle (charge) dynamics by the Newton equation and that of the EM field by Maxwell equations. The semiclassical theory implies that Newton equation is replaced by the Schroedinger equation. The use of prefix "standard" for

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classical theory means that there is another formulation in which the basic ingredients of classical theory are retained, but the emphasis is shifted from the concept of classical trajectory to the probability. For the purpose of this paper this alternative description is of no consequence, and will not be discussed. Only to mention that, for example, the frequency dependent energy transfer, which will be mentioned shortly, can be explained by classical theory [12].

As mentioned earlier, the Compton and the photoelectric effects come together in the scattering of the EM wave on a bound charge, and hence any alternative description should be general enough to account for, if not all, but most of their features. This is precisely the weakness of the standard classical description: while the attempts to explain the frequency shift in the Compton scattering show some success [13–18] the theory is unable to explain the photoelectric effect, or the intensity of the scattered radiation in both of them. In particular the photoelectric effect presents an insurmountable problem for standard classical theory, because it cannot explain the evidence that the energy transfer from the EM plane wave to the charge is a linear function of the frequency. In the classical picture the charge acquires, in the first order of coupling, oscillations in unison with the electric component of the EM field. It is only in the second order of coupling that the charge acquires a uniform velocity, in addition to the velocity which it had before the onset of the interaction. Classical description of the frequency shift for the scattered radiation is based on the Dopler shift which arises from these two effects. However, for the photoelectric effect there is nothing which indicates that the change in the velocity of the charge depends on the frequency of the EM field, as the experimental evidence showed. This fact alone was sufficient to dismiss the standard classical theory for description of the phenomena on the scale of elementary charges. In fact, as we know today, both types of the energy transfer are present (in the extreme of very strong fields the amplitude, *i.e.* classical, energy transfer dominates [19]), but for the low intensity radiation the dominant is the frequency dependent.

Two amendments were suggested, which did not alter the basic concepts of classical theory but accommodate the experimental evidence. They are mentioned because even today the discussion concerning them is used as an argument against alternatives to the concept of the photon. One amendment is due to Einstein, and suggests that the charge absorbs only a "bundle of the EM energy", and not any value. This "bundle of energy" is proportional to the frequency of the EM radiation. As a consequence, the charge acquires energy which is proportional to the frequency of the EM field, but for that certain time is required. In other words, there is a delay between the time of incidence of the EM wave on the charge and the time when it moves. The lower the intensity the larger this time delay. The other amendment suggested that the EM field is transmitted by the photons (particles) whose energy and momentum are proportional to the frequency of the EM wave. The flux of photons in the EM wave is proportional to the squared modules of the EM field (Poynting vector). Being a particle, the photon transfers energy instantaneously and hence there is no time delay. Experiments were made to distinguish between the two amendments and showed that there was no time delay. This result is cited as a positive evidence against classical (and even semiclassical) models, and supports the photon concept. It should be said, though, that this argument is false because the amendment based on the "bundle of energy" is not a workable model which can be put into the framework of classical dynamics. It is more of a "wishful thinking" because there is no way it can be connected with either Newton or Maxwell equations, *i.e.* the two equations put together would not produce result which is envisaged. However, the amendment based on the photon concept is of that kind, *i.e.* by abandoning the EM field equation for the sake of the classical particle equation the correct answer is obtained. Therefore, putting that argument against classical theory misses the point, and the strongest is: the standard classical theory fails because it cannot explain the frequency dependent energy transfer between the EM field and the charge.

The semiclassical theory is more successful for describing interaction between the EM field and the elementary charges, but there are objections to it which need to be discussed. These objections can be put into two groups: those which are based on the "classical" arguments and those which are based on the consequences associated with the effect of the "vacuum polarization". In the second group are the effects such as the Lamb shift or the anomalous magnetic moment of electron. This group of objections will not be discussed here (for a good review see Ref. [20]), but should be pointed out that there were attempts to accommodate them by reformulating the radiation theory so that the concept of the classical EM field is retained. Among these are: random electrodynamics [21] or stochastic electrodynamics (for a review of the subject see Ref. [22]), neoclassical electrodynamics field theory [23] and self field quantum electrodynamics [24,25] (in fact not a proper semiclassical theory). If these objections are neglected we assume tacitly that semiclassical theory: (i) has limitations which are only important on the scale on which we are not working, or (ii) has not been sufficiently investigated so that these objections can be alleviated (it was recently shown that the anomaly in the magnetic moment of electron can be explained without the QED [26], and the work is in progress to show the same for the Lamb shift).

We will briefly discuss the "classical" arguments against the semiclassical theory. Among them is the one which was already discussed: the question of the time delay. Description of the photoelectric effect by semiclassical theory can be found in almost any text book on quantum theory, and some of its more detailed features have been investigated by the same approach [27]. No problem with the time delay was encountered, although this feature was not explicitly investigated. Hence it can be safely assumed that the photoelectric (or ionization) process is instantaneous, in the semiclassical description.

The other argument against the semiclassical theory is based on the energy conservation consideration. If radiation of a very high frequency impacts on an atom then electron of a very high energy is ejected, even for a very weak intensity of radiation. It appears that this is an energy imbalance between the input and the output, if the EM field is assumed. This is indeed the case if it is forgotten that the total energy conservation law assumes the energy of the field and the average energy of the electron [28], and not its precise energy, as in the standard classical theory [29]. The average energy of electron involves probability of finding it with certain energy, which is proportional to the squared amplitude of the EM field. In the model with the photon, on the other hand, it is one thing to analyze energy transfer for the scattering event but completely other to ask what the probability for this event is. The probability involves also the flux of photons, which is proportional to the squared modules of the EM field amplitude. The total energy, therefore, in the semiclassical and the photon model is the same.

There is one additional problem with the semiclassical description of the photoelectric effect. If the photon model is correct then the prediction is that there will be no scattered one because all its energy goes into that of the electron. In the semiclassical picture this means that there would be no scattered radiation. This is shown in the paper, and the qualitative argument which supports the affirmative answer is the following: in the first order of coupling the bound electron is promoted into the continuum, and acquires uniform velocity. In the second order of coupling the free electron oscillates and reradiates the EM field. Therefore, in the photoelectric effect, being of the first order in coupling, there will be no reradiated EM field because electron in the uniform motion does not radiate.

While the semiclassical theory does not have too many problems with the photoelectric effect, it has problems with the Compton effect. Although the frequency shift could be obtained (see *e.g.* Ref. [30]), the biggest problem is explaining the correlation experiments. Briefly: in the Compton effect there is a definite experimental evidence that the frequency of the scattered radiation at a particular angle and the momentum of the scattered charge (electron) are correlated according to the standard expression derived from the photon model. The result of this experiment indicated the failure of the semiclassical radiation theory because it could not explain that the two events are correlated.

In this paper we show that the above potential failures are not present in the semiclassical radiation theory. If that is the case then one can ask a question: in what way then the approach in this paper differs from that which has been described previously, and showed failures? The only difference is in the strict implementation of the classical theory of radiation (reviewed briefly in Sect. 2), and the rules for solving quantum dynamics equations. It is precisely these points which the previous analyses lack, *e.g.* the dipole approximation in the calculation of the spectrum is widely used. As the result of the strict implementation of the classical radiation theory complicated integrals will be encountered. However, by a suitable transformation they can be calculated almost analytically.

In addition to proving previously mentioned points, we also give expression for the Compton cross section and show that it differs fundamentally from the one which is derived from the model based on the concept of the photon [2,6]. Instead of being proportional to the integral over the momentum probability distribution (the Compton profile) in the semiclassical radiation theory it is proportional to the squared modules of the same integral but over the probability amplitude in the momentum space. Crudely speaking, the difference is as between the classical addition of probabilities (in the photon model) and the quantum addition of amplitudes (in the semiclassical radiation theory). Despite this the two cross sections do not differ much in the numerical value, but sufficiently so that the experiment, in principle, would be able to distinguish it. The exception is for the low frequencies of the scattered radiation on the bound charge where we obtain a spectrum which is finite. It is known that in the photon model it is infinite, resulting from what is called the "infrared catastrophe" [31–33], and its usual explanation is in terms of the classical theory of radiation [34]. In this paper the finite spectrum is derived from the semiclassical theory, without having to use additional arguments for distinguishing the low frequency from the high frequency scattered radiation field.

In addition to those features we also obtain the cross section for the inverse Compton scattering. For the localized wave packets, whether free or bound, the effect accompanies the usual Compton scattering.

2 Radiation associated with a charge

The starting point of our analysis is the classical expression for the intensity of radiation produced by a time varying and localized current $\mathbf{J}(\mathbf{r},t)$ [29]

$$\begin{aligned}\mathbf{P} =& \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\\ \approx& \frac{\hat{\mathbf{n}}}{r^2}\left\{\left[\int \mathrm{d}^3\mathbf{r}'\dot{\mathbf{J}}\left(\mathbf{r}',\tau'\right)\right]^2 - \left[\hat{\mathbf{n}}\int \mathrm{d}^3\mathbf{r}'\dot{\mathbf{J}}\left(\mathbf{r}',\tau'\right)\right]^2\right\}\end{aligned} \tag{1}$$

where $\tau' = t - \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{c}$ is the retarded time (the term r/c has been omitted). The unit vector $\hat{\mathbf{n}}$ points in the radial direction, and the dot designates the time derivative.

For the current $\mathbf{J}$ we assume the following properties. Prior to $t = 0$ it is zero, and thereafter it results from interaction between a localized charge distribution (probability distribution) and a linearly polarized EM wave. It will be assumed that the distribution is localized in the space having the approximate dimensions of a hydrogen atom, but otherwise it is free. In a separate section we will discuss the case when the electron is bound in a hydrogen atom.

If the interaction time is T then the total power in the direction $\hat{\mathbf{n}}$ is

$$\mathbf{W} \;=\; \int_0^T \mathrm{d}t'\, \mathbf{P}(\mathbf{r},t) \tag{2}$$

and by taking the Fourier transform of the current

$$\mathbf{J}(\mathbf{r},t) \;=\; \int \mathrm{d}\omega \mathbf{j}\,(\mathbf{r},\omega)\,\mathrm{e}^{-i\omega t} \tag{3}$$

the spectrum $I(\omega)$ of radiation is

$$\begin{aligned} I(\omega) \;&= w^2 \int \mathrm{d}^3\mathbf{r}'\,\mathrm{d}^3\mathbf{r}''\,\{\boldsymbol{j}\,(\mathbf{r}',w)\cdot\mathbf{j}^{\,*}\,(\boldsymbol{r}'',\omega) \\ &- \left[\hat{\mathbf{n}}\cdot\boldsymbol{j}\,(\mathbf{r}',w)\right]\left[\hat{\mathbf{n}}\cdot\mathbf{j}^{\,*}\,(\boldsymbol{r}'',\omega)\right]\}\,\mathrm{e}^{i\frac{\omega\hat{\mathbf{n}}}{c}\cdot\left(\mathbf{r}''-\mathbf{r}'\right)} \end{aligned} \tag{4}$$

This current is the result of an interaction between the wave packet and the EM wave

$$\mathbf{A} \;=\; \hat{\mathbf{x}}\,A_0\,f(z-ct) \;=\; A_0\,\mathbf{a} \tag{5}$$

and depending on the choice of the method for its calculation we distinguish classical and semiclassical approach. In the latter one uses as the tool quantum theory, hence the name semiclassical theory because the spectrum is calculated from the rules of classical electrodynamics. The use of classical theory for calculating the current was made possible only recently, when a suitable amendment was made in it [35]. It was shown on various examples that both the relativistic [36,37] and non relativistic [12] dynamics of a charge in an EM field can be reproduced in almost total accord with quantum theory. Therefore, we could have taken either of the approaches, however, the semiclassical one is relatively easier to implement and this is the only reason why it is used. It should be kept in mind, though, that the same results could be obtained from classical theory.

In this work we will use relativistic quantum theory for calculating the current for an unbound particle, and if for simplicity a scalar (spin 0) particle is considered then the equation which needs to be solved is the Klein-Gordon (KG)

$$\frac{\partial^2\psi}{\partial t^2} \;=\; \left(\boldsymbol{\nabla} - i\epsilon\mathbf{a}\right)^2\psi \;-\; \psi. \tag{6}$$

Its general solution, for the positive energy states only, is given by [36]

$$\psi \;=\; \int \mathrm{d}^3\mathbf{k}\,A(\mathbf{k}\,)\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}-iet+i\delta} \tag{7}$$

where $e=\sqrt{1+k^2}$ and

$$\delta \;=\; -\frac{\epsilon}{2(k_z-e)}\int_{u_0}^{z-t}\mathrm{d}u\,\left[\epsilon f^2(u) \;-\; 2k_x f(u)\right]. \tag{8}$$

The amplitude $A(\mathbf{k}\,)$ is determined from the initial condition ψ_0. Throughout this work we use units in which $\hbar=c=m=1$.

The probability current for the KG equation is

$$\mathbf{J} \;=\; \mathrm{Im}\,(\psi^*\,\boldsymbol{\nabla}\psi) \;-\; \mathbf{a}\,\psi^*\psi \tag{9}$$

and if it is assumed that ϵ is small, and $|e-k_z|\approx 1$ (a non relativistic distribution of momenta), then the current is

$$\begin{aligned} \mathbf{J} =&\mathrm{Im}\,(\psi_0^*\boldsymbol{\nabla}\psi_0) \;+\epsilon\left[\mathrm{Im}\,(\psi_1^*\boldsymbol{\nabla}\psi_0)\right. \\ &\left.+\mathrm{Im}\,(\psi_0^*\boldsymbol{\nabla}\psi_1) - \mathbf{a}\,\psi_0^*\psi_0\right] \end{aligned} \tag{10}$$

where

$$\psi_0 \;=\; \int \mathrm{d}^3\mathbf{k}\,A(\mathbf{k}\,)\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}-iet} \tag{11}$$

and

$$\psi_1 \;=\; \int_{u_0}^{z-t}\mathrm{d}u\,f(u)\int \mathrm{d}^3\mathbf{k}\frac{k_x}{k_z-e}A(\mathbf{k}\,)\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}-iet}. \tag{12}$$

If the initial ψ is spherically symmetric then the first term in (10) has only a radial component and does not produces a radiation field. The remaining terms produce a radiation field but this is quite complicated. However, the current simplifies if the initial ψ_0 is relatively broad, say it corresponds roughly to the dimensions of the 1s state of the hydrogen atom. In this case the spread of momenta is small (non relativistic), meaning that $|A|$ is negligible for k larger than $k_{\max} << 1$. For the 1s state of hydrogen atom the amplitude $A(k)$ drops by a factor α (the fine structure constant) for $k_{\max}\approx\alpha^{3/4} << 1$, which is indeed small compared to 1. The terms with ψ_1 in (10) can therefore be neglected and so the current is

$$\mathbf{J} \;=\; -\epsilon\,\hat{\mathbf{x}}\,\psi_0^*\psi_0\,f(z-t). \tag{13}$$

If $f(z-t)=\cos\left[w_0(z-t)\right]$ then the Fourier transform of the current, which enters in (4), is

$$\begin{aligned} \mathbf{j}\,(\mathbf{r},w) =&\frac{i\epsilon\hat{\mathbf{x}}}{8\pi}\int \mathrm{d}^3\mathbf{k}\,\mathrm{d}^3\mathbf{k}'\,A(\mathbf{k}\,)A^*(\mathbf{k}')\mathrm{e}^{i\mathbf{r}\cdot\left(\mathbf{k}-\mathbf{k}'\right)} \\ &\times\left[\frac{\mathrm{e}^{iT\left(\omega-e+\mathrm{e}'-\omega_0\right)}-1}{\omega-e+\mathrm{e}'-\omega_0}\mathrm{e}^{i\omega_0 z}+(\omega_0\to-\omega_0)\right] \end{aligned} \tag{14}$$

and the spatial integral over $\mathbf{j}(\mathbf{r},\omega)$ in (4) is

$$\begin{aligned} \int \mathrm{d}^3\mathbf{r}\,\mathrm{e}^{-i\omega\hat{\mathbf{n}}\cdot\mathbf{r}}\mathbf{j}\,(\mathbf{r},\omega) \approx \hat{\mathbf{x}}\int \mathrm{d}^3\mathbf{k}\,A(k)A(k_1) \\ \times\frac{\mathrm{e}^{iT(\omega-e+e_1-\omega_0)}-1}{\omega-e+e_1-\omega_0} \;+\; (\omega_0\to-\omega_0) \end{aligned} \tag{15}$$

where $\mathbf{k}_1=\mathbf{k}+\omega_0\hat{\mathbf{z}}-\omega\hat{\mathbf{n}}$ and $e_1=\sqrt{1+k_1^2}$. In deriving this it has been assumed that $A(k)$ is real and depends only on the modulus of $\mathbf{k}$.

3 Spectrum of a free charge

To understand the spectrum, the integral (15) should be calculated, when the parameter T is large. More precise definition of large T will be discussed in the following section, while for the moment it is assumed to be infinite. The integral (15) can then be evaluated approximately by using the variable k_z, which is defined as the projection of the vector $\mathbf{k}$ on $\boldsymbol{\Delta} = \omega\hat{\mathbf{n}} - \omega_0\hat{\mathbf{z}}$, *i.e.*

$$\mathbf{k}\cdot\boldsymbol{\Delta} \;=\; k_z\Delta. \tag{16}$$

The integral which needs to be calculated is therefore

$$B = \int_{-\infty}^{\infty} \mathrm{d}k_z\, A(k)A(k_1)\frac{\mathrm{e}^{iT(\omega-e+e_1-\omega_0)}-1}{\omega-e+e_1-\omega_0} + (\omega_0\to-\omega_0). \tag{17}$$

In the first step the integration path is slightly shifted into the lower half of the complex k_z plane, where $\mathrm{Im}(-e+e_1) < 0$ and hence $\exp[iT(-e+e_1)]$ can be neglected. Without the neglected term the integrand in (17) has a pole when

$$\omega \;-\; e \;+\; e_1 \;-\; \omega_0 \;=\; 0 \tag{18}$$

and is given by

$$k_z^0 = \frac{1}{2}|\omega\hat{\mathbf{n}} \;-\; \omega_0\hat{\mathbf{z}}| + \frac{\omega-\omega_0}{2}\left[\frac{2+2\kappa^2+\omega\omega_0(1-n_z)}{\omega\omega_0(1-n_z)}\right]^{\frac{1}{2}} \tag{19}$$

where $\kappa^2 \;=\; k_x^2 + k_y^2$. By returning the integration path back to the real k_z axes the contribution of this pole should be taken into account. The integral is therefore

$$B \;=\; -\,i\pi\frac{A\left(k^0\right)A\left(k_1^0\right)\mathrm{e}^0e_1^0}{k_z^0(\omega-\omega_0)-\mathrm{e}^0\Delta} \;-\; \mathrm{P}\int_{-\infty}^{\infty}\mathrm{d}k_z\,\frac{A(k)A(k_1)}{\omega-\omega_0-e+e_1} \;+\; (\omega_0\to-\omega_0) \tag{20}$$

where the superscript 0 designates that k_z is replaced by k_z^0. The label P indicates the principal value of this integral.

The first term in (20) is the contribution from the pole. It has a maximum value for either $k_z^0 = 0$ or $k_z^0 - \Delta = 0$ because of the assumption that $A(k)$ is localized in the narrow interval around $|\mathbf{k}| = 0$. Each equation has only one solution for ω. The solution of the first is designated by ω_c while that of the second by ω_ic. They are given by

$$\omega_\mathrm{c} \;=\; \frac{\omega_0}{1+\omega_0(1-n_z)} \quad ; \quad \omega_\mathrm{ic} \;=\; \frac{\omega_0}{1-\omega_0(1-n_z)} \tag{21}$$

where κ^2 has been neglected.

If $\omega_0 < 1$ and $\omega_0 \approx \omega >> k^2$ then k_z^0 is approximately

$$k_z^0 \;\approx\; \frac{\Delta}{2} \;+\; \frac{\omega-\omega_0}{\sqrt{2\omega\omega_0(1-n_z)}} \tag{22}$$

where the estimates $\omega - \omega_0 \approx \omega_0^2$ and $\Delta \approx \omega_0$ have been utilized, and it follows that

$$k_z^0(\omega_0-\omega) \;+\; \mathrm{e}^0\Delta \;\approx\; \mathrm{e}^0\Delta \tag{23}$$

Furthermore, k_1 is nearly independent of κ^2 because $\omega - \omega_0 \approx \omega_\mathrm{c} - \omega_0 \approx \omega_0^2$ and $\Delta \approx \omega_0$, hence the contribution in (20) from the poles ω_c and ω_ic is approximately

$$B_\mathrm{p} \approx i\pi\frac{A(\Delta)\sqrt{1+\Delta^2}}{\Delta} \times\left[A\left(\sqrt{\kappa^2+k_z^{0^2}}\right) \;+\; A\left(\sqrt{\kappa^2+\left(k_z^0-\Delta\right)^2}\right)\right]. \tag{24}$$

The contribution in (20) from the principal value of the integral can also be estimated. If it is assumed that $k_z^0 \approx 0$, *i.e.* ω is close to the frequency ω_c, then one can show that this contribution is negligible. When k_z^0 is large, *i.e.* $|k_z^0| >> \alpha$, then we can write

$$\mathrm{P}\int_{-\infty}^{\infty}\mathrm{d}k_z\frac{A(k)A(k_1)}{\omega-\omega_0-e+e_1} \approx \int_{-\eta}^{\eta}\mathrm{d}k_z\cdots + \int_{\Delta-\eta}^{\Delta+\eta}\mathrm{d}k_z\cdots \tag{25}$$

where η is small but sufficiently large so that $|A(\eta)|$ is negligible. Within the limits of integration the integrands are nearly constant except for $A(k)$ in the first integral and $A(k_1)$ in the second. When this is taken into account the integral (20) is approximately

$$B = \frac{(\omega-\omega_0)A(\Delta)}{1+\omega\omega_0(1-n_z)-\sqrt{1+\Delta^2}}\int_{-\infty}^{\infty}\mathrm{d}k_z\,A(k) + i\pi\frac{A(\Delta)\sqrt{1+\Delta^2}}{\Delta} \times\left[A\left(\sqrt{\kappa^2+k_z^{0^2}}\right) \;+\; A\left(\sqrt{\kappa^2+\left(k_z^0-\Delta\right)^2}\right)\right]. \tag{26}$$

The difference between the exact B, given by (17), and (26) is shown in Figure 1 for $\omega_0 = 0.1$, and for two scattering angles. The real and imaginary parts of B are shown separately. For the amplitude $A(k)$ the momentum distribution for a 1s wave function has been taken. The exact value of the integral (17) is shown by a solid line while the approximation (26) by a dotted line. The agreement for both the imaginary and the real parts is very good, for all ω, except in a small vicinity of ω_c and ω_ic, where this is not the case for the real part. However, in this region the exact real part is relatively small compared to the imaginary part, and hence it can be neglected.

By using the approximation (26) the integral (15) is now

$$\int\mathrm{d}^3\mathbf{r}\cdots \approx\; \hat{\mathbf{x}}\frac{4\pi(\omega-\omega_0)A\left(\Delta\right)}{1+\omega\omega_0(1-n_z)-\sqrt{1+\Delta^2}}\int_0^{\infty}\mathrm{d}k\,k^2A(k) + \hat{\mathbf{x}}\,\frac{2\pi^2iA\left(\Delta\right)\sqrt{1+\Delta^2}}{\Delta} \times\left[\int_{|k_z^0|}^{\infty}\mathrm{d}p\,pA(p) \;+\; \int_{|k_z^0-\Delta|}^{\infty}\mathrm{d}p\,pA(p)\right] \tag{27}$$

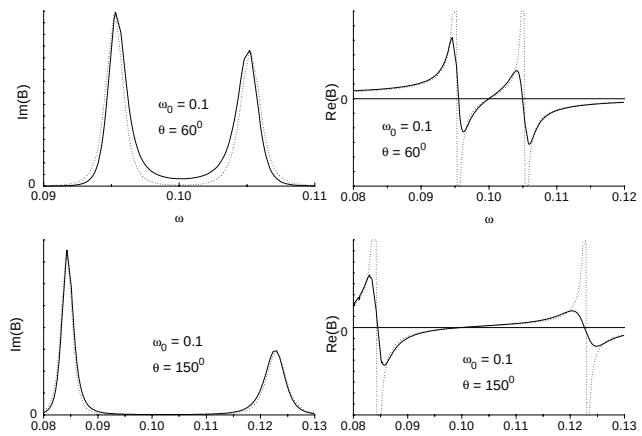

Fig. 1. Real and imaginary parts of the Fourrier transform of the spatial integral over the current for the scattering of EM wave on a wave packet which has the form of the 1s wave function of a hydrogen atom. The solid line represents the exact value and the dotted line represents the approximation derived in the text.

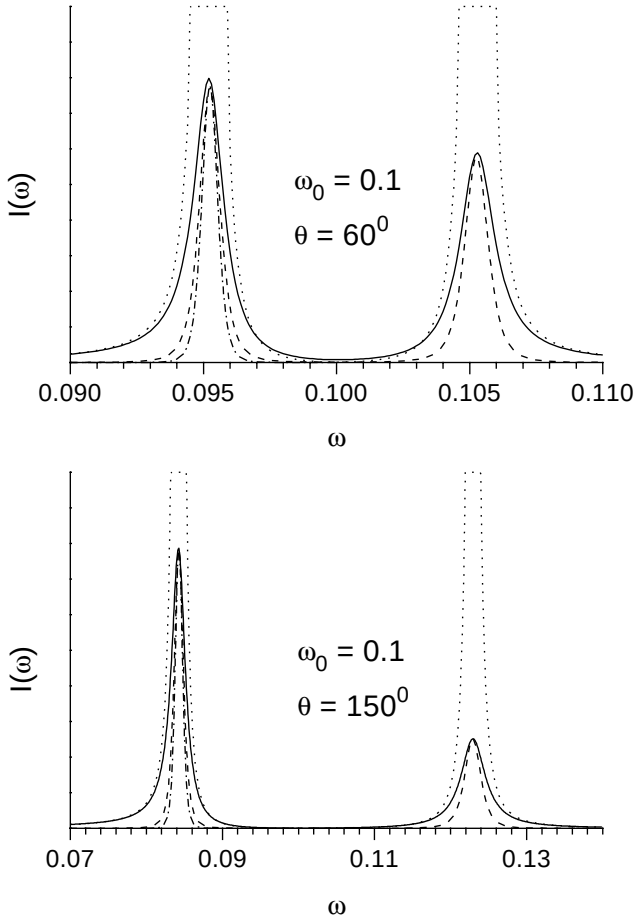

Fig. 2. Spectrum resulting from the scattering of an EM wave on a free wave packet which has the shape of the 1s wave function of hydrogen atom. The solid line represents the exact spectrum, the dotted line is the contribution from the real part of the integral B, and the broken line is its imaginary part. Comparison is made with the spectrum based on the model by DuMond (dotted-broken line). The spectrum is in arbitrary units with a common normalization factor.

which is used for calculating the spectrum given by (4). One typical is shown in Figure 2, for the example treated in Figure 1, where the solid line represents the spectrum which is calculated from (20). The dotted and the broken lines show separately the contribution from the real and imaginary parts of (27). The two contributions approximate the spectrum in two different intervals. The contribution from the pole (imaginary part of (20)) approximates the spectrum in the vicinity of $\omega_{\rm c}$ or $\omega_{\rm ic}$, but the contribution from the principal value of the integral in (20) (real part of B) approximates the spectrum outside those regions.

In the vicinity of $\omega_{\rm c}$ the spectrum takes a simple form if the approximation (27) is used; it is given by

$$I(\omega) \approx \omega^2 \frac{A^2(\Delta)\left(1+\Delta^2\right)}{\Delta^2}\left[\int_{|k_z^0|}^{\infty} \mathrm{d}p\, A(p)\right]^2 \left[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{z}})^2\right]. \tag{28}$$

A characteristic feature of the spectrum is that it consists of two contributions: (i) the factor $A^2(\Delta)$ which is the probability of finding an electron with the momentum $-\mathbf{\Delta}$ (in the derivation of (26) the amplitude A is a function of $|-\mathbf{\Delta}|$) and (ii) the remainder which can be interpreted as the "bare" contribution to the radiation intensity. In other words, the intensity of radiation is weighted by the probability of finding an electron with the momentum $\mathbf{k}_1 = -\mathbf{\Delta}$. This shows that the dominant frequency of radiation, which is scattered in the angle θ, is correlated with the momentum of the electron. It is interesting that if one formally associates $\omega_{\rm c}\hat{\mathbf{n}}$ with the momentum of the scattered radiation and $\omega_{\rm c}$ with its energy, then the equations which determine the relationship between these two correlated parameters are

$$\begin{aligned} \mathbf{k} + \omega_0\hat{\mathbf{z}} &= \mathbf{k}_1 + \omega\hat{\mathbf{n}} \\ e + \omega_0 &= e_1 + \omega. \end{aligned} \tag{29}$$

The first equation is obtained when evaluating (15) and the second when evaluating (17). In these equations we recognize the momentum and energy conservation law for the system consisting of an electron and a (hypothetical) particle (we can call it a photon). The first has momentum $\mathbf{k}$ and energy e while the photon carries momentum $\boldsymbol{\omega}$ and energy ω. The frequency $\omega_{\rm c}$, given by (21), is obtained by taking initial momentum of electron $\mathbf{k} = 0$ and initial energy $e = 1$, while for the photon these quantities are $\omega_0\hat{\mathbf{z}}$ and ω_0, respectively. The index c of ω derives from the work of Compton, because he originally used the concept of the photon to explain the frequency shift of the scattered radiation [38]. This relationship, as we have seen in this paper, can be derived without introducing this concept.

The frequency $\omega_{\rm ic}$ is derived from the equations

$$\begin{aligned} \mathbf{k} + \omega_0 \hat{\mathbf{z}} &= \omega_{\rm ic}\, \hat{\mathbf{n}} \\ e + \omega_0 &= 1 + \omega_{\rm ic} \end{aligned} \tag{30}$$

where it has been assumed that $\mathbf{k}_1 = 0$. The meaning of these is that the photon is scattered by the electron which carries momentum $\mathbf{k}$ and after the event the electron is stopped, *i.e.* $\mathbf{k}_1 = 0$. This effect is known and it is called the inverse Compton (ic) scattering. In the case when the electron is entirely delocalized (plane wave) it is not observed together with the Compton scattering, as it is in our case. It is observed as a separate event, but for this a very energetic electron is needed, such is in the cosmic processes [39]. However, for a localized electron there is a spread of momenta and therefore the two effects are observed together, the cross section for the inverse Compton scattering being always weaker. When we will consider bound electron they will always be present.

The approximate spectrum (28) should be correlated with the one which is most widely accepted, which goes back to the intuitive reasoning of DuMond [2], and summarized many times (*e.g.* Ref. [5]). It should be mentioned that we are dealing with a free wave packet and therefore the impulsive approximation, from which the DuMond cross section is derived, is nearly exact. There is a big difference between the two spectra (the cross sections) and briefly this can be summarized as follows: in the approach described here the spectrum is proportional to the square of the integral over $A(k)$, while in DuMond's formula it is proportional to the integral over $A^2(k)$; specifically, the spectrum in DuMond's treatment is proportional to

$$I(\omega) \;\approx\; \frac{\omega}{\Delta} \int_{|k_z^0|}^{\infty} {\rm d}p\, A^2(p) \tag{31}$$

and for the sake of comparison with the treatment here its value is shown in Figure 2 by the broken-dotted line. In the vicinity of $\omega_{\rm c}$ it is quite close to the contribution (28). However, this does not eliminate the fundamental difference between the treatment based on the photon model, the result of which is given essentially in the form (31), and the one based on the semiclassical radiation theory, the result of which is (28). As mentioned in Introduction, the result is as between classical (photon model) and quantum theory (semiclassical theory). In the first one sums the probabilities and in the second one sums the amplitudes, and then takes the squared modulus.

4 The low frequency limit

The problem with the derivation in Section 3 is that the limit $T \to \infty$ was taken formally, without considering physical implications. The result is a spectrum $I(\omega)$ which does not have the correct limit for $\omega_0 \to 0$, when it should approach the so called Thompson cross section, *i.e.*

$$\lim_{\omega_0 \to 0} I(\omega) \;\approx\; \left[1 + \cos^2(\theta)\right]. \tag{32}$$

This is obviously not the case with (28) (and also the cross section (31)), where the angle dependence is of the form $(1 + \cos^2(\theta))/(1 - \cos(\theta))$ (after averaging over the polarization of the EM wave). As we shall see, in order to satisfy this limit it is essential to assume that T is finite, but large.

In the analysis of Section 3 the limit $T \to \infty$ was in fact not essential. It was sufficient to assume in (17) that on the integration path in the lower half of the complex k_z plane the function $\exp\left[iT(-e + e_1)\right]$ is negligible. The question is only how negative the imaginary part of k_z should be before this approximation can be made. If the width of $A(k)$ is of the order α then it is reasonable to assume that this shift cannot be larger than this value. For example, the momentum space 1s wave function of a hydrogen atom is

$$A(k) \;\approx\; \frac{1}{\left(\alpha^2 + k^2\right)^2} \tag{33}$$

which has a pole at $k_z = \pm i\alpha$ (for $\kappa = 0$). If the shift in k_z is beyond this pole, because T is not sufficiently large, then the contribution to the integral (17) comes from the pole at $-i\alpha$, and B is proportional to

$$B \;\approx\; \frac{{\rm e}^{iT(\omega - \omega_0 + i\alpha\Delta)} - 1}{\omega - \omega_0 + i\alpha\Delta} \tag{34}$$

instead of having the value (26).

The previous discussion gives a natural criterion for deciding what is large T. If ${\rm e}^{-T\alpha\omega_0} << 1$ or $T >> 1/(\alpha\omega_0)$ (we used the approximation $\Delta \approx \omega_0$) then T is large. Based on this criterion it is evident that the derivation in Section 3 is not legitimate because the assumption $T \to \infty$ was made regardless of the value ω_0. A proper derivation, therefore, would require that T has a large, but a finite value, and then the integral (17) is calculated. If the criterion for large T is not fullfiled then it cannot be calculated by the described method. The same reasoning also applies when the width of the momentum distribution shrinks, or equivalently when the particle becomes delocalized in a wider space. In this respect the results of Section 3 are also not correct in the plane wave limit ($\alpha \to 0$).

When T is not large, meaning that $T\alpha\omega_0 << 1$, the integral (17) is approximately

$$B \;\approx\; \frac{{\rm e}^{iT(\omega - \omega_0)} - 1}{\omega - \omega_0} \tag{35}$$

and the spectrum takes a very simple form

$$I(\omega) \;\approx\; \left[1 - \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}\right)^2\right] \delta(\omega - \omega_0) \tag{36}$$

which is the functional form of the Thompson radiation intensity. The identity

$$\lim_{T \to \infty} \left[\frac{1}{T} \frac{{\rm e}^{iT(\omega - \omega_0)} - 1}{\omega - \omega_0} \frac{{\rm e}^{-iT(\omega - \omega_0)} - 1}{\omega - \omega_0}\right] \;=\; 2\pi\delta\left(\omega - \omega_0\right) \tag{37}$$

and approximation $k_1 \approx k$ was used in the derivation.

5 Scattering of an EM wave by a bound charge

In the previous discussion we have considered the scattering of a plane EM wave on a free, but localized, charge. Here we will discuss the more realistic case when this charge is bound and initially in a stationary (say the ground) state n_0. The EM plane wave is given by (13) and the solution of Schroedinger equation is given in the form of expansion

$$\psi \;=\; \sum_n a_n(t)\,\phi_n(\mathbf{r}\,) \;+\; \int \mathrm{d}k\, b_k(t)\,\Phi_k(\mathbf{r}\,) \tag{38}$$

where for simplicity the index n and the integral over k also assumes a sum over the angular momentum states. If it is assumed $\omega_0 >> \alpha >> |E_{n_0}|$ then the dominant coefficient is

$$b_k^{(1)} \;=\; \mathrm{e}^{-E_k t}\int_0^t \mathrm{d}t'\,\mathrm{e}^{it'\left(E_k - E_{n_0}\right)} < k|\mathbf{a}\cdot\boldsymbol{\nabla}|n_0 > \tag{39}$$

and the leading term in the current is

$$\mathbf{J}^{\,(1)} \;=\; \mathrm{Im}\left[\psi^{(1)^*}\boldsymbol{\nabla}\psi^{(0)} + \psi^{(0)^*}\boldsymbol{\nabla}\psi^{(1)}\right] \;-\; \mathbf{A}\,|\psi^{(0)}|^2 \tag{40}$$

where $\psi^{(0)} = \phi_{n_0}(\mathbf{r}\,)\mathrm{e}^{-iE_{n_0}t}$, and $\psi^{(1)}$ is given by (38) where the coefficient is (39). The contribution of the second term in (40) to the spectrum is relatively simple. The spatial integral over the Fourier transform of the current is

$$\int \mathrm{d}^3\mathbf{r}\,\mathrm{e}^{-i\omega\hat{\mathbf{n}}\cdot\mathbf{r}}\,\mathbf{j}\,(\mathbf{r},\omega) \approx \frac{\hat{\mathbf{x}}}{2}\int \mathrm{d}^3\mathbf{r}|\psi^{(0)}|^2\mathrm{e}^{-i\omega\hat{\mathbf{n}}\cdot\mathbf{r}\,+\,i\omega_0 z}\,\frac{\mathrm{e}^{iT(\omega-\omega_0)}-1}{i(\omega-\omega_0)} \tag{41}$$

which in the spectrum contributes a single line at $\omega = \omega_0$, irrespective of the scattering angle. This term, therefore, represents elastic scattering of radiation, in contrast to the radiation by a free distribution where it contributes to Compton scattering. An explanation for the origin of Compton scattering is now simple. A free wave packet has a natural tendency to spread; this is the time evolution of the probability distribution. This is coupled with the time evolution due to the interaction with the external field leading to a frequency shift of the scattered radiation. The unperturbed bound states, by their definition, are stationary and their time evolution is in the first approximation entirely determined by the external field, and hence there is no frequency shift of radiation.

The resulting radiation is all concentrated in the forward direction because the momentum distribution is narrow and the only contribution to the integral (41) comes from the region where $\hat{\mathbf{n}}\cdot\mathbf{r} \approx z$.

The contribution to the spectrum arising from the first term in (40) can be extracted from the integral

$$\int \mathrm{d}^3\mathbf{r}\,\mathrm{e}^{-i\omega\hat{\mathbf{n}}\cdot\mathbf{r}}\int \mathrm{d}t\,\psi^{(0)^*}\boldsymbol{\nabla}\psi^{(1)}\,\mathrm{e}^{i\omega t} \;= -\frac{1}{2}\int \mathrm{d}k\,\frac{< n_0|\mathrm{e}^{-i\omega\hat{\mathbf{n}}\cdot\mathbf{r}}\boldsymbol{\nabla}|k >< k|\mathrm{e}^{i\omega_0 z}\boldsymbol{\nabla}_x|n_0 >}{E_k - E_{n_0} - \omega_0} \times\left[\frac{\mathrm{e}^{iT(\omega-\omega_0)}-1}{\omega-\omega_0} - \frac{\mathrm{e}^{iT(\omega+E_{n_0}-E_k)}-1}{\omega+E_{n_0}-E_k}\right]. \tag{42}$$

From the assumption that ω_0 is large the function $\Phi_k(\mathbf{r}\,)$ can be replaced by a plane wave $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$. The matrix elements in (42) are then given explicitly as

$$< n_0|\mathrm{e}^{-i\omega\hat{\mathbf{n}}\cdot\mathbf{r}}\boldsymbol{\nabla}|k > \approx \frac{\mathbf{k}}{\left[\alpha^2 + \left(\mathbf{k}-\omega\hat{\mathbf{n}}\,\right)^2\right]^2} = \mathbf{k}\,A\left(|\mathbf{k}-\omega\hat{\mathbf{n}}\,|\right). \tag{43}$$

Evaluation of the integral in k space follows the same procedure as in Section 3. The integration path in k_z is slightly shifted into the lower half plane where the function e^{-iTE_k} can be neglected. The remaining terms of the integrand (42) have poles at $E_k = E_{n_0}+\omega_0$ and $E_k = E_{n_0}+\omega$. When the path is returned back to the real k_z axes, and shifted into the upper half of the k_z plane, the pole at $k_z^0 = \sqrt{2(E_{n_0}+\omega_0) - k_x^2 - k_y^2}$ contributes

$$\int \mathrm{d}^3 r\,\cdots \;\approx\; \int \mathrm{d}k_x\,\mathrm{d}k_y\,\frac{\mathbf{k}_0 k_x}{k_0}A\left(|\mathbf{k}_0-\omega\hat{\mathbf{n}}\,|\right) \times A\left(|\omega_0\hat{\mathbf{z}}-\mathbf{k}_0\,|\right)\,\frac{\mathrm{e}^{iT(\omega-\omega_0)}-1}{\omega-\omega} \tag{44}$$

where $\mathbf{k}_0$ has the z component equal to k_z^0. The integrals in k_x and k_y have their maximum contributions at $k_x = k_y = 0$ and $n_x = n_y = 0$, so that (44) is negligible because of the presence of the factor k_x in the integrand.

The small intensity of the scattered radiation has physical meaning. The spectrum arising from the first term in the current (40) consists of a single line at $\omega = \omega_0$, but the electron comes out with the energy $E_{n_0} + \omega_0$. This term, therefore, describes the spectrum of radiation resulting from the photoelectric effect, and its negligible value is consistent with the usual explanation: a photon of energy ω_0 is absorbed by the electron and there is no scattered photon but electron is emitted with energy $\omega_0 + E_{n_0}$. It should be reemphasized: in our treatment the intensity of radiation was obtained from the semiclassical radiation theory without using the concept of the photon.

6 The second order correction

The correction to the current which is of second order in ϵ is

$$\mathbf{J}^{\,(2)} = \mathrm{Im}\left[\psi^{(2)^*}\boldsymbol{\nabla}\psi^{(0)} \;+\; \psi^{(0)^*}\boldsymbol{\nabla}\psi^{(2)}\right] \;-\; \mathbf{a}\left(\psi^{(1)^*}\psi^{(0)} \;+\; \psi^{(0)^*}\psi^{(1)}\right) \tag{45}$$

and only the spectrum arising from the last term will be analyzed because the contribution of the first is negligible. The integral (41) for this part of the current is

$$\int \mathrm{d}^3\mathbf{r}\cdots = \hat{\mathbf{x}}\int \mathrm{d}^3\mathbf{r}\,\mathrm{e}^{-i\omega\hat{\mathbf{n}}\cdot\boldsymbol{r}} \times \int_0^T \mathrm{d}t\,\mathrm{e}^{i\omega t}\cos\left[\omega_0(z-t)\right]\left[\psi^{(0)^*}\psi^{(1)} + \psi^{(1)^*}\psi^{(0)}\right] \tag{46}$$

and if $\psi^{(1)}$ is replaced by its explicit form we get a very lengthy expression which can be put in the form

$$\int \mathrm{d}^3\mathbf{r}\cdots \approx \int \mathrm{d}^3\mathbf{k}\,k_x\,[I_1 + I_2] \tag{47}$$

where

$$\begin{aligned} I_1 = &\Big[\frac{A(\mathbf{k}-\omega\hat{\mathbf{n}}+\omega_0\hat{\mathbf{z}})A(\mathbf{k}-\omega_0\hat{\mathbf{z}})}{E_k-\omega_0-E_{n_0}} \\ &\times\left(\frac{\mathrm{e}^{iT(\omega-2\omega_0)}-1}{\omega-2\omega_0}-\frac{\mathrm{e}^{iT(\omega+E_{n_0}-E_k-\omega_0)}-1}{\omega+E_{n_0}-E_k-\omega_0}\right) \\ &+(\omega_0\to-\omega_0)\Big] \\ &-(E_k\to-E_k,\ E_{n_0}\to-E_{n_0},\ \mathbf{k}\to-\mathbf{k}) \end{aligned} \tag{48}$$

and

$$\begin{aligned} I_2 = &\Big[\frac{A(\mathbf{k}-\omega\hat{\mathbf{n}}+\omega_0\hat{\mathbf{z}})A(\mathbf{k}+\omega_0\hat{\mathbf{z}})}{E_k+\omega_0-E_{n_0}} \\ &\times\left(\frac{\mathrm{e}^{iT\omega}-1}{\omega}-\frac{\mathrm{e}^{iT(\omega+E_{n_0}-E_k-\omega_0)}-1}{\omega+E_{n_0}-E_k-\omega_0}\right) \\ &+(\omega_0\to-\omega_0)\Big] \\ &-(E_k\to-E_k,\ E_{n_0}\to-E_{n_0},\ \mathbf{k}\to-\mathbf{k}). \end{aligned} \tag{49}$$

The arrows indicate how the variables in the previous term should be replaced by new ones. In the next step the integration is distorted either into the lower or the upper half of the complex k_z plane, depending on the sign of E_k. On this path the function $\mathrm{e}^{\pm iTE_k}$ in the integrand can be neglected, and for the remaining terms the path is shifted back to the real k_z axes. The integral (47) will now have a form very similar to (20); part of it is the contribution from the poles and part of it is from the principal value of the integral. In the analysis here the principal value of the integral will not be discussed in detail. There are a number of poles in the variable k_z, but among them only three basic ones; (i) the pole at $E_k = \omega_0 + E_{n_0}$ represents the photoelectric effect, (ii) the pole $E_k = \omega_0 + E_{n_0} - \omega$ represents Compton scattering and (iii) the pole $E_k = \omega - \omega_0 + E_{n_0}$ represents the inverse Compton scattering. It can be shown that the contributions of most of the poles are negligible and only two remain; that from the Compton effect and that from the inverse Compton effect. The pole contribution (imaginary part of (46)) from the Compton effect is given by

$$\begin{aligned} \mathrm{Im}\left[\int \mathrm{d}^3\mathbf{r}\cdots\right] \approx \pi\sum_{s=0}^{1}(-1)^s \int \mathrm{d}k_x\,\mathrm{d}k_y\,(k_x-\omega n_x) \\ \times\frac{A(\mathbf{k}_\mathrm{c}^s)}{\sqrt{2\,(E_{n_0}+\omega_0-\omega)-k_x^2-k_y^2}} \\ \times\left[\frac{A(-\mathbf{k}_\mathrm{c}^s-\omega\hat{\mathbf{n}}+2\omega_0\hat{\mathbf{z}})}{\omega-2\omega_0}+\frac{A(-\mathbf{k}_\mathrm{c}^s-\omega\hat{\mathbf{n}})}{\omega}\right] \end{aligned} \tag{50}$$

where $\mathbf{k}_\mathrm{c}^s = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z^s\hat{\mathbf{z}}$ and

$$k_z^s = |\omega\hat{\mathbf{n}}-\omega_0\hat{\mathbf{z}}| + (-1)^s\sqrt{2(\omega_0+E_{n_0}-\omega)-k_x^2-k_y^2}. \tag{51}$$

Most of this contribution comes from the vicinity of $|k_\mathrm{c}^s| = 0$, which is only possible when $k_z^1 = 0$. This condition gives the value of ω, designated by ω_c, for which the maximum intensity in the spectrum is expected. The frequency ω_c has more meaning if it is expanded in the series of ω_0 (energy E_{n_0} is treated as being of the order ω_0^2), where the first three terms are the same as in the expansion of the function

$$\omega_\mathrm{c} = \frac{\omega_0+E_{n_0}}{1+\omega_0(1-n_z)} \tag{52}$$

which is the well known Compton frequency shift function (in the relativistic derivation) modified, however, by the bound state energy E_{n_0}. The source of this shift is explained by noting that ω_c is a solution of the same set of equations as (29), where the initial momentum of electron ($\mathbf{k}$) is set to zero, and its initial kinetic energy to $E_k = E_{n_0}$ instead of to zero (in the non relativistic treatment e is replaced by E_k). This modification of the frequency shift of the scattered radiation, in the case of a bound electron, should not be confused with the so called "Compton defect". The latter is the name for the deviation between the exact and the approximate expression for the Compton profile [6].

In the vicinity of ω_c the integral (50) is approximately

$$\begin{aligned} \mathrm{Im}\left[\int \mathrm{d}^3\mathbf{r}\cdots\right] \approx 2\pi^2\omega n_x\left[\frac{A(-\omega\hat{\mathbf{n}}+2\omega_0\hat{\mathbf{z}})}{\omega-2\omega_0}+\frac{A(-\omega\hat{\mathbf{n}})}{\omega}\right] \\ \times\int_{|k_z^1|}^{k_z^0}\mathrm{d}p\,p\,\frac{A(p)}{\sqrt{2\,(E_{n_0}+\omega_0-\omega)}} \end{aligned} \tag{53}$$

where we set $k_x = k_y = 0$ in k_z^s. This integral is different from the analogous one for the free wave packet (27) by the factor which represents the probability amplitude of the electron momentum. However, it can be shown from ψ^1 that the factor in front of the integral (53) has the same meaning as for a free wave packet, and hence the two expressions are in a total accord.

A similar derivation can be used in the vicinity of ω_ic but it is not discussed here.

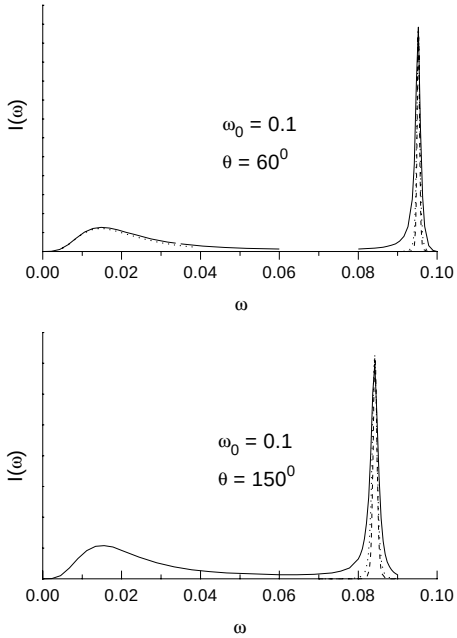

Fig. 3. The same as Figure 2, but for a bound charge in the 1s state of hydrogen atom.

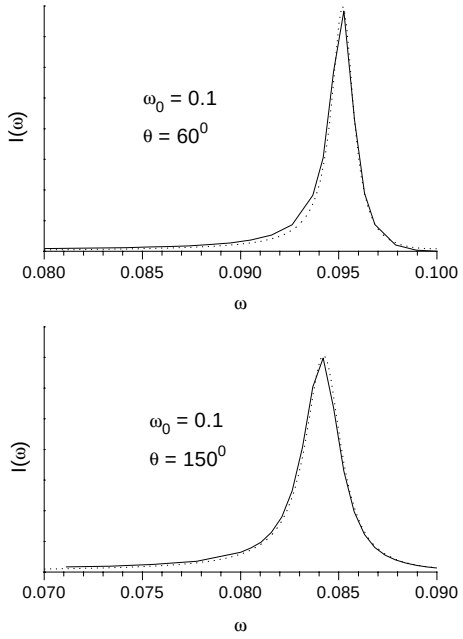

Fig. 4. Comparison of spectra (exact) from the bound (solid line) and the free (dotted line) charge.

An example has been calculated with the same parameters as used for Figure 1. The spectrum is shown in Figure 3, where the integral (47) has been calculated numerically (solid line). For comparison we also show the contribution from the pole term in (47) (the dotted line - the approximation (53) is indistinguishable from the numerical calculation of the pole term in (47)). The spectrum based on the impulsive model (31) is also shown (broken line). More accurate cross section in the photon model is not used because the deviation from the impulsive model, for the incident frequency of radiation which is used here, is marginal [4]. As expected, the contribution (53) approximates well the spectrum in the immediate vicinity of ω_c, where also the spectrum (31) gives nearly the same result. Away from the vicinity of ω_c the pole term alone is not sufficient to reproduce the exact spectrum which is calculated from the integral (47).

Additional structure is observed in the low frequency limit. This is the region where in the photon model one gets infinite cross section, and it is called the "infrared catastrophe". Therefore it is of interest to give more details of its source in the semiclassical theory. The contribution to the integral (47) for $\omega \approx 0$ comes largely from the term I_2, because the product $|A(\mathbf{k}+\omega_0\hat{\mathbf{z}})A(\mathbf{k}+\omega_0\hat{\mathbf{z}})|$ is always larger than $|A(\mathbf{k}+\omega_0\hat{\mathbf{z}})A(\mathbf{k}-\omega_0\hat{\mathbf{z}})|$ in the term I_1. This also means that most of the contribution comes from the vicinity of $\mathbf{k} = \pm\omega_0\hat{\mathbf{z}}$ because it is assumed that the amplitude A is relatively narrow and peaks at $k = 0$. For large T the oscillatory terms can be omitted (the exponents do not have stationary points) in I_2 and $E_k \approx \omega_0^2/2$ (and E_{n_0}) can be neglected. In this approximation the spectrum which arises from this process is given by

$$I(\omega) \;\approx\; \frac{w^2}{w_0^4}\left|\int \mathrm{d}^3\mathbf{k}\, k_x\, A(\mathbf{k})\,[A(\mathbf{k}-\omega\hat{\mathbf{n}}) - A(\mathbf{k}+\omega\hat{\mathbf{n}})]\right|^2 \tag{54}$$

which is independent of ω_0, appart in the prefactor. In Figure 3 the approximate spectrum in the low frequency part is shown by the dotted line (not visible for $\theta = 150°$). There is no simple physical explanation for this part of the spectrum. Simple neglect of the oscillatory part of I_2 was possible because there is no contribution from the poles, and hence the entire spectrum comes from the background term. As discussed previously, it is the pole term which determines the nature of the energy transfer, while the background term corrects the magnitude of the spectrum. Further study of this part of the spectrum is required.

Although scattering of an EM wave by free and bound charges are two different processes, and the resulting spectra have different expressions, their numerical values are nearly the same (after appropriate normalization). In Figure 4 the spectra of the Figures 2 and 3 are combined together. The spectrum resulting from scattering by a bound charge is shown by a solid line and that of a free charge by a dotted line. The two are nearly indistinguishable, thus supporting the conjecture resulting from the photon model. In the high photon energy regime the charge behaves as a free particle and hence the two cross sections should be the same. In this paper we showed this from first principles, without using the photon concept. However, this conclusion only applies in the vicinity of the frequency ω_c, and not in the low frequency interval. The additional peak in Figure 3 for small frequencies is not reproduced by a free charge.

7 Conclusion

We have shown how to analyze the spectrum which results from interaction of an EM wave and a charge, either free or bound, by using semiclassical radiation theory. It was assumed that the free charge was localized in a finite space, in our treatment this was taken as the 1s state of a hydrogen atom. For a free charge we have used the relativistic treatment, based on the Klein-Gordon equation, a choice which was primarily motivated by its simplicity. A treatment which uses the Dirac equation is along the same lines, but it is somewhat more complicated because of the additional contributions resulting from the spin of the electron. Otherwise, the technical details of the derivations and their discussion are the same as for the Klein-Gordan equation, because the solution for the wave function is know exactly [37]. The result should be the same as from the well known treatment which is based on the photon model [1].

We have shown that semiclassical theory reproduces all the known results from the Compton scattering, including the most difficult conceptual problems of the correlation experiments. The latter are usually cited as a drawback of semiclassical radiation theory. The spectrum has also been calculated and shown to be different from the one derived from the photon model. The difference is essentially of the same kind as the difference between classical and quantum theory.

References

1. O. Klein, Y. Nishina, Z. Phys. **52**, 853 (1929).
2. J.W.M. DuMond, Rev. Mod. Phys. **5**, 1 (1933).
3. P.M. Bergstom, T. Surić, K. Pisk, R.H. Pratt, Phys. Rev. A **48**, 1134, (1993).
4. B.J. Bloch, L.B. Mandelsohn, Phys. Rev. A **9**, 129 (1974).
5. M.J. Cooper, Rep. Prog. Phys. **48**, 415 (1985).
6. P. Eisenberg, P.M. Platzman, Phys. Rev. A **2**, 415 (1970).
7. A. Issolak, B. Levy, A. Beswick, G. Loupias, Phys. Rev. A **38**, 4509, (1988).
8. H.A. Bethe, *Intermediate Quantum Mechanics* (Benjamin, 1964) chapt. 15.
9. E. Fermi, Rev. Mod. Phys. **4**, 87 (1932).
10. H.A. Kramers, W. Heisenberg, Z. Phys. **31**, 681 (1925).
11. P.A.M. Dirac, Proc. R. Soc. London A **114**, 243 (1927).
12. N. Doslic, S.D. Bosanac, Z. Phys. D **32**, 261 (1995).
13. J.N. Dodd, J. Phys. B **8**, 157 (1975).
14. D.G. Ashworth, R.C. Jennison, J. Phys. A **7**, 803 (1974).
15. R. Kidd, J. Ardini, A. Anton, Am. J. Phys. **53**, 641 (1985).
16. W.R. Mellen, Am. J. Phys. **49**, 505 (1981).
17. E. Schroedinger, Ann. Phys. **82**, 257 (1927).
18. O. Halpern, Z. Phys. **30**, 153 (1924).
19. Y.I. Salamin, F.H.M. Faisal, Phys. Rev. A **54**, 4383 (1996).
20. M.O. Scully, M. Sargent III, Physics Today **25**, 38 (1972).
21. T.H. Boyer, Phys. Rev. D **11**, 790 (1975).
22. T.H. Boyer, in *Survey of Stochastic Electrodynamics* in *Foundations of Radiation Theory and Quantum Electrodynamics*, edited by A.O. Barut (Plenum, 1980) p. 49.
23. J.H. Eberly, *Unified View of Spontaneous Emission*, *ibid.*, p. 23.
24. A.O. Barut, J. Kraus, Y. Salomin, N. Üval, Phys. Rev. A **45**, 7740 (1992).
25. A.O. Barut, J.F. Van Huele, Phys. Rev. A **32**, 3187 (1985).
26. S.D. Bosanac, *Non-QED theory of the anomalous magnetic moment of an electron* (submitted for publication).
27. S.D. Bosanac, Z. Phys. D **37**, 103 (1996).
28. S.D. Bosanac, Phys. Rev. A **50**, 2899 (1994).
29. J.D. Jackson, *Classical Electrodynamics* (Wiley, 1975).
30. J. Barwick, Phys. Rev. A **17**, 1912 (1978).
31. W. Heitler, *The Quantum Theory of Radiation* (Dover, 1984).
32. F.E. Low, Phys. Rev. **110**, 974 (1958).
33. T.H. Burnet, N.M. Kroll, Phys. Rev. Lett. **20**, 86 (1968).
34. A.I. Akhiezer, V.B. Berestetskii, *Quantum Electrodynamics*, (Interscience, 1965).
35. H. Skenderovic, S.D. Bosanac, Z. Phys. D **35**, 107 (1995).
36. S.D. Bosanac, Z. Phys. D **24**, 325 (1992).
37. S.D. Bosanac, J. Phys. A **26**, 5523 (1993).
38. A.H. Compton, Phys. Rev. **21**, 483 (1923); **22**, 409 (1923).
39. M. Harwit, *Astrophysical Concepts* (Springer, 1988) p. 243.